Given two normed space X and Y, a linear map $T: X \to Y$ is cts. iff its bounded.

- $\|T\| := \sup\{ \|Tx\|_Y : x \in X \text{ with } \|x\|_X \le 1 \}$
  $= \inf\{ C : \|Tx\|_Y \le C\|x\|_X \text{ for all } x \in X \}$

$\|Tx\|_Y \le \|T\| \cdot \|x\|_X$

Examples :

1. Let $X = (C[0,1], \|\cdot\|_\infty)$. $S, T : X \to X$

$(Tf)(x) = x \cdot f(x)$

$(Sf)(x) = x \cdot \int_0^1 f(y)\,dy$.

Find $\|T\|$, $\|S\|$, $\|T \circ S\|$, $\|S \circ T\|$.

Ans:

- Claim: $\|T\| \le 1$

$(\|Tf\|_\infty \le C\|f\|_\infty, \ \forall f, \Rightarrow \|T\| \le C)$

$\|Tf\|_\infty = \|x \cdot f(x)\|_\infty = \max_{x \in [0,1]} |x f(x)| \le 1 \cdot \|f\|_\infty$

$\Rightarrow \|T\| \le 1$.

- Claim: $\|T\| \ge 1$.

$(\exists f_0 \in X, \ \|Tf_0\|_\infty \ge C_2 \|f_0\|_\infty, \Rightarrow \|T\| \ge C_2)$

Take $f \equiv 1$, and then

$\|Tf\|_\infty = \max_{x \in [0,1]} |x \cdot 1| = 1 = \|f\|_\infty$ $(\|f\|_\infty = 1)$

$\|T\| \ge 1$

Therefore, $\|T\| = 1$.

$$(Tf)(x) = x\cdot f(x)$$
$$(Sf)(x) = x\cdot \int_0^1 f(y)\,dy$$

② Similarly,

$$\|Sf\|_\infty = \max_{x\in[0,1]} \left| x \int_0^1 f(y)\,dy \right| = 1\cdot \int_0^1 f(y)\,dy$$
$$\le \max_{y\in[0,1]} f(y)\cdot(1-0) = \|f\|_\infty$$

$\|S\| \le 1$.

Take $f\equiv 1$, $\|S\| \ge 1$.

③ $$STf(x) = S(Tf)(x) = S(g)(x) = x\cdot\int_0^1 g(y)\,dy$$
$$g(x) := Tf(x) = xf(x) \qquad = x\int_0^1 y f(y)\,dy$$

$$\|STf\|_\infty = \max_{x\in[0,1]} |STf(x)| = \left|\int_0^1 y f(y)\,dy\right|$$
$$\le \int_0^1 y\cdot|f(y)|\,dy$$
$$\le \int_0^1 y\cdot \max_{y\in[0,1]}|f(y)|\,dy$$
$$= \int_0^1 y\,\|f\|_\infty\,dy = \|f\|_\infty \tfrac{1}{2}$$

That is, $\|STf\|_\infty \le \frac{1}{2}\cdot\|f\|_\infty$

$\Rightarrow \|ST\| \le \frac{1}{2}$.

Take $f\equiv 1$, $STf(x) = S(Tf)(x) = S(g)(x) = x\cdot\int_0^1 g(y)\,dy$
$g(x) = Tf(x) = x\cdot f(x) = x$, $\qquad = x\cdot\int_0^1 y\,dy$

$$STf(x) = x\cdot\int_0^1 y\,dy$$
$$\|STf\|_\infty = \max_{x\in[0,1]} \left|x\int_0^1 y\,dy\right| = 1\cdot\tfrac{1}{2} = \tfrac{1}{2}\cdot\|f\|_\infty$$
$$\Rightarrow \|ST\| \ge \tfrac{1}{2}$$

Therefore, $\|ST\| = \frac{1}{2}$.

$$(Tf)(x) = x \cdot f(x)$$
$$(Sf)(x) = x \cdot \int_0^1 f(y)\,dy$$

④ $TSf(x) = T(Sf)(x) = x \cdot h(x) = x^2 \cdot \int_0^1 f(y)\,dy$

$h(x) = Sf(x) = x \cdot \int_0^1 f(y)\,dy$

$\|TSf\|_\infty = \max_{x \in [0,1]} \left| x^2 \cdot \int_0^1 f(y)\,dy \right| = \|f\|_\infty$

$\Rightarrow \|TS\| \le 1$

Take $f \equiv 1$, $\|TS\| \ge 1$.

Therefore, $\|TS\| = 1$.

Remark: $\cdot\ \|S \circ T\| \ne \|T \circ S\|$

$\cdot\ \|S \circ T\| \le \|S\| \cdot \|T\|$

(pf: $\|S \cdot T(f)\| \le \|S\| \cdot \|Tf\|_\infty \le \|S\| \cdot \|T\| \cdot \|f\|_\infty$

for any $f \in X$.

2. Let $I=[a,b]$, $Tf(x)=\int_a^x f(t)\,dt$.

Then $T\in B(L^1(I), C[I])$, and

$T\in B(L^1(I), L^1(I))$.

Find $\|T\|$ for each case.

- $\|T\| := \sup\{\|Tx\|_Y : x\in X \text{ with } \|x\|_X\le 1\}$

$= \inf\{C : \|Tx\|_Y \le C\|x\|_X \text{ for all } x\in X\}$

Ans: $(C[I], \|\cdot\|_\infty)$, $(L^1(I), \|\cdot\|_1)$.

$T(f)\in C[I]$

Considering $T\in B(L^1(I), C[I])$.

$(\exists C.\ \|Tf\|_\infty \le C\|f\|_1 \Rightarrow \|T\|\le C)$

$$\|Tf\|_\infty = \max_{a\le x\le b}\left|\int_a^x f(t)\,dt\right| \le \max_{a\le x\le b}\int_a^x |f(t)|\,dt$$

$$= \int_a^b |f(t)|\,dt = 1\cdot\|f\|_1$$

$\|T\|\le 1$.

Take $f\equiv 1$, then $\|f\|_1=\int_a^b 1\,dx = b-a$

$Tf(x) = \int_a^x 1\,dt = x-a$

$\|Tf\|_\infty = \max_{x\in[a,b]}|x-a| = b-a = 1\cdot\|f\|_1$

$\Rightarrow \|T\|\ge 1$

Considering $T\in B(L^1(I), C[I])$, then $\|T\|=1$

Considering $T \in B(L^1(I), L^1(I))$.

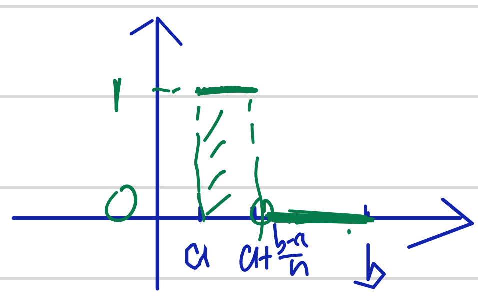
$\Big(\overset{\exists C.}{\|Tf\|_1} \le C\|f\|_1 . \Rightarrow \|T\| \le C \Big)$

$$\begin{aligned}
\|Tf\|_1 &= \int_a^b |Tf(x)|\,dx = \int_a^b \left|\int_a^x f(t)\,dt\right| dx \\
&\le \int_a^b \int_a^x |f(t)|\,dt\,dx \\
&= \iint \mathbb{1}_{\{(t,x):\, a<t<x<b\}} |f(t)|\,dt\,dx \\
&\overset{\text{Fubini thm}}{=} \int_a^b \int_t^b |f(t)|\,dx\,dt \\
&= \int_a^b (b-t)\,|f(t)|\,dt \\
&\le \int_a^b (b-a)\,|f(t)|\,dt = (b-a)\cdot\|f\|_1
\end{aligned}$$

$\|Tf\|_1 \le (b-a)\cdot\|f\|_1 \Rightarrow \|T\| \le b-a$

Claim: $\|T\| \ge b-a$.

$$f_n(t) = \begin{cases} 1, & a < t \le a+\frac{b-a}{n} \\ 0, & a+\frac{b-a}{n} < t < b \end{cases}$$



$\|f_n\|_1 = \frac{b-a}{n}, \quad \forall n \in \mathbb{N}$.

$$Tf_n(x) = \int_a^x f(t)\,dt = \begin{cases} x-a & a \le x \le a+\frac{b-a}{n} \\ \frac{b-a}{n} & a+\frac{b-a}{n} < x \le b \end{cases}$$

$$\|Tf_n\|_1 = \int_a^b \left|\int_a^x f_n(t)\,dt\right| dx = \frac{(b-a)^2}{n} - \frac{(b-a)^2}{2n^2} = \frac{b-a}{n}\left[(b-a) - \frac{b-a}{n}\right]$$

$$= \|f_n\|_1 \cdot \left((b-a) - \frac{b-a}{n}\right)$$

$\Rightarrow \|T\| \ge (b-a) - \frac{b-a}{n}$, let $n \to \infty$, $\|T\| \ge b-a$

Therefore, $\|T\| = b-a$

Remark: When regarded as operators on different normed space, the operator norm may differ.

3 For a infinite column vector $X = (x_1, x_2, \cdots)^T$, define $Ax = (2x_1, x_2 - x_1, x_3 - x_2, \cdots)$. Then $A \in B(l^1, l^1)$ and $A \in B(l^\infty, l^\infty)$. Find $\|T\|$ for each case.

Ans: $l^p = \{ (x_i)_i : (\sum_i |x_i|^p) < \infty \}$, $p \geq 1$

$l_\infty = \{ (x_i)_i : \sup_i (x_i) < \infty \}$

$A : l^1 \to l^1$

$$\|Ax\|_1 = |2x_1| + |x_2 - x_1| + |x_3 - x_2| + \cdots,$$
$$\leq 2|x_1| + (|x_1| + |x_2|) + (|x_2| + |x_3|) \cdots$$
$$= 3|x_1| + 2|x_2| + 2|x_3| + \cdots$$
$$= |x_1| + 2 \sum_{i=1}^{\infty} |x_i|$$
$$\leq \|x\|_1 + 2 \cdot \|x\|_1 \leq 3 \cdot \|x\|_1$$

Thus, $\|A\| \leq 3$.

Take $x = (1, 0, 0, \cdots 0)$, $\|x\| = 1$

$Ax = (2, -1, 0, \cdots 0)$

$\|Ax\|_1 = 2 + |-1| = 3 = 3 \cdot \|x\|_1$

$\Rightarrow \|A\| \geq 3$.

Therefore, $\|A\| = 3$.

$A : l_\infty \to l_\infty$,

$$\|Ax\|_\infty = \sup\{ |2x_1|, |x_2 - x_1|, \cdots \} \leq \underline{\sup\{ 2|x_1|, |x_2| + |x_1|, \cdots \}}$$
$$\underline{\leq 2 \sup\{ |x_1|, |x_2|, \cdots \}} = 2 \cdot \|x\|_\infty$$

if $|x_2| + |x_1| \leq \sup\{2|x_1|, 2|x_2|$

$\Rightarrow \|A\| \leq 2$.

Take $x = (1, 0, \cdots 0)$.

we can see that $\|A\| \geq 2$.